University, Statesboro, GA, and the proposer.

- **5291:** *Proposed by Arkady Alt, San Jose, CA*

Let $m_a m_b$ be the medians of a triangle with side lengths a, b, c. Prove that:

$$m_a m_b \leq \frac{2c^2 + ab}{4}.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

We wish to prove that

$$\begin{aligned} 2c^2 + ab - 4m_a m_b &\geq 0 \text{ or equivalently,} \\ \left(2c^2 + ab + 4m_a m_b\right)\left(2c^2 + ab - 4m_a m_b\right) &\geq 0, \text{that is,} \\ \left(2c^2 + ab\right)^2 - 16m_a^2 m_b^2 &\geq 0. \end{aligned}$$

Since $m_a = \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$, and $m_b = \dfrac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$ we obtain:

$$(2c^2 + ab)^2 - 16m_a^2 m_b^2 = \left(2c^2 + ab\right)^2 - \left(2b^2 + 2c^2 - a^2\right)\left(2c^2 + 2a^2 - b^2\right)$$

$$\begin{aligned} &= 4c^4 + 4abc^2 + a^2b^2 - \left(4b^2c^2 + 4a^2b^2 - 2b^4 + 4c^4 + 4c^2a^2 - 2b^2c^2 - 2c^2a^2 - 2a^4 + a^2b^2\right) \\ &= 4abc^2 - 4a^2b^2 - 2b^2c^2 - 2c^2a^2 + 2a^4 + 2b^4 \\ &= 2a^4 + 2b^4 - 4a^2b^4 - 2b^2c^2 - 2c^2a^2 + 4abc^2 \\ &= 2\left(\left(a^2 - b^2\right)^2 - (bc - ca)^2\right) \\ &= 2\left((a+b)^2(a-b)^2\right) - c^2(b-a)^2 \\ &= 2(a-b)^2\left((a+b)^2 - c^2\right) \\ &= 2(a-b)^2(a+b+c)(a+b-c) \geq 0 \end{aligned}$$

By the triangle inequality $a + b - c > 0$, with equality if and only if $a = b$, that is , if and only if the triangle is isosceles with equal side lengths a and b.

Solution 2 by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain

Since the length of the medians of any triangle ABC with side lengths a, b, and c are given by the expression

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \text{(cyclic)},$$

as it is well-known, then the inequality claimed becomes

$$\left(\frac{1}{2}\sqrt{2b^2+2c^2-a^2}\right)\left(\frac{1}{2}\sqrt{2c^2+2a^2-b^2}\right)\leq\frac{2c^2+ab}{4}$$

or

$$\sqrt{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)}\leq 2c^2+ab$$

Squaring both sides of the above inequality and after canceling terms, we obtain

$$2a^4+2b^4-4c^2ab-4a^2b^2-2b^2c^2-2c^2a^2\geq 0$$

or equivalently,

$$2(a-b)^2(a+b+c)(a+b-c)\geq 0$$

which is true on account that in any non degenerate triangle ABC is $a+b>c$. Equality holds when $a=b$. That is when $\triangle ABC$ is isosceles, and we are done.

Also solved by D. M. Bătinetu-Giurgiu,"Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania, and Titu Zvonaru, Comănesti, Romania; Ed Gray, Highland Beach, FL; Kenneth Korbin, New York, NY; Paul M. Harms, North Newton, KS, Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Ecole Suppa, Teramo, Italy,and the proposer.

- **5292:** *Proposed by D.M. Bătinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

Let a and b be real numbers with $a<b$, and let c be a positive real number. If $f: R\longrightarrow R_+$ is a continuous function, calculate:

$$\int_a^b \frac{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}}{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}+e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}dx.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

If $f(x)=e^{f(x-a)}(f(x-a))^{\frac{1}{c}}$ and $g(x)=e^{f(b-x)}(f(b-x))^{\frac{1}{c}}$, then for $x\in(a,b)$, $f(x)=g(b-x+a)$ and hence the proposed integral, say I is equal to

$$I=\int_a^b \frac{e^{f(b-x)}(f(b-x))^{\frac{1}{c}}}{e^{f(x-a)}(f(x-a))^{\frac{1}{c}}+e^{f(b-x)}(f(b-x))^{\frac{1}{c}}}\, dx,$$

and so $I=\dfrac{b-a}{2}$.

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

By letting $y=\dfrac{x-a}{b-a}$, the integral is equal to

$$I=(b-a)\int_0^1 \frac{F((b-a)y)}{F((b-a)y)+F((b-a)(1-y))}dy$$